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Space-Time Isomorphism Problem is Intractable (NP-Hard)

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The problem of whether or not two different mathematical models of space-time describe the same space-time is not trivial. For example, the first spherically symmetric solution—the Schwarzschild metric—describes only part of the process of falling into a black hole, and other metrics were discovered for the same situation that also describe the subsequent events. These metrics turned out to be isomorphic in the sense that some 1–1 correspondence (coordinates transformations) transform one into another. But in the general case to find whether such an isomorphism exists is a difficult computational problem. There are some algorithms for smooth metrics, but the problem is also important for the non-smooth metrics involving singularities. We prove that in the general case this isomorphism problem is intractable.

1. INTRODUCTION

When physicists analyze problems in space-time physics, they solve the corresponding equations and often find several solutions. The equations for space-time geometry are difficult, so there is no routine way to find solutions; therefore physicists try to apply new ideas, e.g., by imposing symmetry or some special algebraic structure [for a survey see, e.g., Kramer *et al.* (1980)]. Different approaches often lead to solutions expressed by different formulas. Therefore, if someone presents a new solution, it is necessary to figure out whether this is really a new space-time model or this new solution is isomorphic to one of the already known ones in the sense that some coordinate transformation turns the first solution into the second one.

This problem dates back to Christoffel and since then several algorithms have been proposed and successfully used (see, e.g., Misner *et al.*, 1973; MacCallum, 1986, 1988; Kamran, 1988), but they are not always applicable. This problem has already been solved for the main solutions of general

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relativity, but in general no efficient algorithm is known (MacCallum, 1988) and this problem still presents a real challenge in case one analyzes alternative theories like the scalar-tensor theory of gravitation, even if one analyzes spherically symmetric solutions (Bharma *et al.*, 1978; Pandey *et al.*, 1983). This "equivalence" problem is conjectured to be difficult to solve (MacCallum, 1988).

In the present paper we prove that this isomorphism problem is in the general case really intractable. Following the usage of computer science (Garey and Johnson, 1979), we call a problem intractable if the possibility of an algorithm that solves this problem in feasible time would lead to the reasonable-time algorithm for solving practically all discrete problems, which is commonly believed to be impossible. The main result of this paper was announced in Kreinovich (1989).

2. HOW TO FORMULATE THIS ISOMORPHISM PROBLEM IN MATHEMATICAL TERMS: DISCUSSION

The first idea that naturally comes to mind is to formulate this problem as looking for an isomorphism, i.e., 1–1 correspondence f between the points of the two given models, such that the distances $d(a, b)$ and $d(fa, fb)$ between the corresponding pairs of points coincide. However, the real physical problem is more complicated.

First of all, when physicists ask whether the two given solutions represent the same space-time or not, then by "the same space-time" they mean not only the possibility to establish a 1–1 correspondence between the points of the corresponding spaces. For example, the original Schwarzschild metric describes only the behavior of the particle falling on a black hole, and the Robertson metric describing also its further expansion into some other space surely describes the same space-time—although in this case 1–1 correspondence does not exist: there is no analog of these expansion-stage points in the original Schwarzschild solution. So in reality when we ask the question about the isomorphism we really have in mind the possibility or impossibility of imbedding one of the models isomorphically into another.

Second, all the existing methods of checking isomorphism are known for the smooth case [see, e.g., the survey by MacCallum (1988)], but this question is also very important near the singularity, where, strictly speaking, the formalism of pseudo-Riemannian geometry is no longer applicable. So it is necessary to formulate this problem (and try to solve it) in the most general way, without assuming that the metrics depend smoothly on coordinates and even—in view of the possible quantum models of space-time—that such coordinates exist at all.

A natural nonsmooth generalization of the notion of a normal (positive-definite) Riemann space is the notion of a metric space: for every Riemann space we can define the distance $d(a, b)$ between two arbitrary points a and b as a minimal length of all the curves connecting a and b. Due to this definition this function $d(a, b)$ satisfies the well-known properties of symmetry $d(a, b) = d(b, a)$, $d(a, a) = 0$, $d(a, b) > 0$ for different a and b, and the triangle inequality that $d(a, c)$ does not exceed $d(a, b) + d(b, c)$ for all a, b, c. Then as a natural generalization of a Riemann space we can consider arbitrary metric spaces, i.e., spaces on which the function d satisfying these conditions is defined.

For pseudo-Riemann spaces the natural analogue of the metric is the proper time $t(a, b)$ between the events a and b, i.e., the maximal proper time along all the timelike curves that connect a and b. This quantity is defined only if a causally precedes b (denoted $a < b$); for the pairs a, b for which no such connection is possible, $t(a, b)$ is formally defined as 0. This function satisfies the following properties:

(i) $t(a, a) = 0$.

(ii) The relation $t(a, b) > 0$ is an ordering relation.

(iii) If $t(a, b) > 0$ and $t(b, c) > 0$, then $t(a, c)$ is not less than $t(a, b) + t(b, c)$.

So a natural generalization of pseudo-Riemann spaces to the nonsmooth case is the space with a function t satisfying these three properties. Such spaces were introduced in Busemann (1967) and Pimenov (1970) (see also Kreinovich, 1974, 1979*a*,*b*). Therefore, we will formulate the isomorphism problem for such spaces.

A special case is when we compare static models, for which it is reasonable to speak about proper space. Then these proper spaces are just metric spaces, and our problem transforms into the following one: to find whether two given metric spaces are isomorphic or not.

Moreover, quantum considerations can lead to space-time models to which the notion of metric is not applicable at all because in the case of strong quantum fluctuations there is no way to measure time at all and hence no sense of speaking about the metric; see, e.g., the last chapters of Misner *et al.* (1973). In these cases the model of space-time is formulated only in terms of a causality relation $<$, that is, from a mathematical viewpoint, just an ordering relation. So we must analyze the problem of isomorphism also for this case.

One last simple, but important remark: we are talking about algorithms; real algorithms stop after finitely many computational steps, during which they can read only finitely many values $d(a, b)$ or $t(a, b)$, so they can read the data only about finitely many points. So in reality any algorithm for checking isomorphism checks the isomorphism of finite sets of events.

The real space-time, however, is normally believed to be infinite. So real algorithms give only approximate answers to the question of whether two given space-time models are equivalent or not (of course, the more points we take into consideration, the more accurate is the result). In view of this, we consider only finite sets.

For the same reason, the computer can process only finitely many digits of the value of each of the distances, so the results of the algorithms are the same as when all distances are binary-rational numbers. Therefore, it is sufficient to consider only the case when all the distances are rational numbers.

3. MATHEMATICAL FORMULATION

Definitions

1. By a *finite metric space* we mean a pair consisting of an integer n and a matrix with nonnegative rational elements $d(a_i, d_j)$, $i, j = 1, 2, \ldots, n$, such that:

(a) $d(a_i, a_j) = 0$ iff $i = j$.

(b) $d(a_i, a_j) = d(a_j, a_i)$ for all i, j.

(c) For all i, j, k the value $d(a_i, a_k)$ is less than or equal to $d(a_i, a_j) + d(a_j, a_k)$.

2. By a *finite space-time model* we mean a pair of an integer n and a matrix with nonnegative rational elements $t(a_i, a_j)$, $i, j = 1, 2, \ldots, n$, such that:

(a) $t(a_i, a_i) = 0$ for all i.

(b) For all i, j, if $t(a_i, a_j) > 0$, then $t(a_j, a_i) = 0$.

(c) If $t(a_i, a_j) > 0$ and $t(a_j, a_k) > 0$, then $t(a_i, a_k)$ is greater than or equal to $t(a_i, a_j) + t(a_j, a_k)$.

3. By a *finite causal model* we understand a pair of an integer n and a matrix $C(a_i, a_j)$, $i, j = 1, 2, \ldots, n$, with elements $1, -1$, and 0 such that:

(a) $C(a_i, a_i) = 0$ for all i.

(b) $C(a_i, a_j) = -C(a_j, a_i)$.

(c) If $C(a_i, a_j) = 1$ and $C(a_j, a_k) = 1$, then $C(a_i, a_k) = 1$.

Comment. $C(a, b) = 1$ means $a > b$, $C(a, b) = -1$ means $a < b$; the conditions (a)–(c) mean that $>$ is an ordering relation.

Definitions

1. By a *mapping* $f\colon A \to A'$ of two finite sets we mean a mapping that assigns to every a_i from A some element $f(a_i) = a_j^*$ from A'.

2. We say that a mapping $f: A \to A'$ is an *isomorphic imbedding* of:
(a) Finite metric spaces, if $d(fa, fb) = d(a, b)$ for all a, b.
(b) Finite space-time models if $t(fa, fb) = t(a, b)$ for all a, b.
(c) Finite causal models if $C(fa, fb) = C(a, b)$ for all a, b.

Comment. The last condition means that $fa < fb$ iff $a < b$.

4. MAIN RESULT

Preliminary Remark. We want to prove that the problem of figuring out whether the two given space-time models can be isomorphically imbedded into each other is intractable. Before we formulate the main result, let us recall what "intractable" means in computer science (Garey and Johnson, 1979). An algorithm is called *feasible* if its running time T does not exceed some polynomial $P(L)$ of the length L of the input data (in this case it is also said that this algorithm finishes its work in polynomial time). If the algorithm is not feasible, then normally its running time is equal to the exponential of L (2^L or something like this), and for $L = 100$ it exceeds the number of particles in the universe—so it is intractable. If we have a discrete problem, i.e., a problem where the possible set of answers is limited to a finite set, then in principle it is possible to solve this problem by analyzing all candidates for a solution. If the possible solutions are all possible binary words of length n, then this lookthrough algorithm must analyze 2^n cases and is therefore intractable in any reasonable sense. Examples of discrete problems comprise all spheres of human activity: in mathematics the problem of proving or disproving a hypothesis H consists in finding a proof of H or $\sim H$ whose length does not exceed some reasonable limit L, so in principle we can find this proof by analyzing all a^L words of length L or less (here a is the number of symbols in our alphabet). In physics the main problem of explaining experimental data can also be in principle solved by enumerating all possible laws and testing whether they fit the experimental data or not. In other words, practically all creative human activity consists in solving discrete problems, so it is highly improbable that there exists an algorithm that can solve all discrete problems in reasonable time: such an algorithm would mean the end to science. So it is widely believed that such an algorithm is impossible.

In the beginning of the 1970s it was proved that some discrete problems have the following properties: if we are able to solve them in polynomial time, then we are able to solve all discrete problems in polynomial time. In view of the above arguments, this means that it is highly improbable that these problems can be solved in polynomial time. Therefore, such problems were called *NP-hard* or *intractable*.

Main Theorem. The following problems are NP-hard:

1. Given two finite metric spaces A, A', to find out whether there exists an isomorphic imbedding $A \to A'$ or not.
2. Given two finite space-time models A, A', to find out whether there exists an isomorphic imbedding $A \to A'$ or not.
3. Given two finite causal models A, A', to find out whether there exists an isomorphic imbedding $A \to A'$ or not.

As a corollary we get the following auxiliary result.

Definition. Let $e > 0$ be any positive real number. A subset B of a metric space A is called *e-distinct* if $d(b, b')$ is greater than or equal to e for all pairs of different points b, b' from B. The *e-capacity* of a metric space A is defined as the binary logarithm of the maximum possible number of elements in its e-distinct subset.

Corollary. The following problems are NP-hard:

1. Given a finite metric space A, a positive integer n, and a positive rational number $e > 0$, to find an e-distinct set with n elements in A.
2. Given a finite metric space A and a positive rational number $e > 0$, to compute the e-capacity of A.

This corollary was first announced in Kreinovich (1979*a*).

Comment. Another example of a physically meaningful NP-hard problem is the prediction problem in quantum mechanics (Kreinovich *et al.*, 1991).

5. PROOFS

Preliminary Remark. A finite causal model, i.e., a finite ordered set, is a particular case of the finite oriented graph. For nonoriented graphs it is known that the problem of whether an isomorphic imbedding exists is NP-hard: namely, the problem of whether it is possible to imbed a complete graph with m elements (so-called "clique") into a given graph G is NP-complete (Garey and Johnson, 1979). We show that this "clique problem" can be reduced to each of our three problems; therefore, the possibility to solve one of them in reasonable (=polynomial) time would lead to the possibility to solve the clique problem in polynomial time, and the clique problem is NP-hard. So this reduction proves that our three problems are also NP-hard.

Let us first show this reduction for finite causal models. Assume G is an arbitrary finite nonoriented graph with the set of vertices V. Let us construct a finite causal model $C(G)$ as follows: its elements are pairs $(v, -1)$, $(v, 0)$, $(v, 1)$, where v is a vertex of G, and $(v, e) < (v', e')$ iff either $v = v'$ and $e < e'$, or $e = -1$, $e' = 1$, and v, v' are connected by an edge in G. We want to show that G can be isomorphically imbedded into G' iff $C(G)$ can be isomorphically imbedded into $C(G')$.

It is easy to prove that if $f: G \to G'$ is an isomorphic imbedding of graphs, then the mapping $f^*: C(G) \to C(G')$, defined as $f^*(v, e) = (f(v), e)$, is an isomorphic imbedding of finite causal models.

Assume now that $f: C(G) \to C(G')$ is an isomorphic imbedding of finite causal models. Let us construct the corresponding imbedding of graphs.

Let us show that f transforms any element $(v, -1)$ into the element of the same type, i.e., into $(v', -1)$ for some v'. Indeed, according to the definition of $C(G)$, we have $(v, -1) < (v, 0) < (v, 1)$. Therefore $f(v, -1) < f(v, 1)$, so there exist elements b, c such that $f(v, -1) < b < c$. But in $C(G')$ such b, c exist only for the elements of the type $(v', -1)$. So $f(v, -1) = (v', -1)$ for some v'. Let us denote this v' by $f^*(v)$. So $(f^*(v), -1) < b = f(v, 0) < c = f(v, 1)$.

From the definition of $C(G)$ it follows that the property $(v', -1) < b < c$ is true only for $b = (v', 0)$ and $c = (v', 1)$; therefore $f(v, 0) = (v', 0)$ and $f(v, 1) = (v', 1)$, hence $f(v, e) = (f^*(v), e)$ for all v, e. It is now easy to check that if f is an isomorphic imbedding of finite causal models, then f^* is an isomorphic imbedding of graphs.

So the problem of whether there exists an imbedding of a clique G_0 into a graph G is equivalent to the problem of whether there exists an imbedding of a finite causal model $C(G_0)$ into $C(G)$. Therefore, the isomorphism imbedding problem for finite causal models is NP-hard.

Let us now prove that the problem of isomorphic imbedding of finite space-time models is NP-hard. To every causal model C we can put into correspondence a finite space-time model $S(C)$ by setting $t(a, b)$ equal to the minimal length of the monotone chain connecting a, b if such a chain exists, and 0 if it does not exist. One can check that all three conditions in the definition of $t(a, b)$ are fulfilled [so $S(C)$ is really a space-time model], and that $C(G) \to C(G')$ is an isomorphic imbedding of the causal models iff it is an isomorphic imbedding of space-time models $S(C(G)) \to S(C(G'))$. So the isomorphism problem for space-time models is also NP-hard.

For finite metric spaces the proof is even simpler. Let us fix some $e > 0$. To every graph G we can put into correspondence the finite metric space $M(G)$ with vertices of G as points and the distance function $d(a, b) = 0$ if $a = b$; $d(a, b) = 4/5e$ if the vertices a, b are not connected in G

and $d(a, b) = e$ if a, b are connected. It is easy to check the triangle inequality and the fact that $f\colon G \to G'$ is an isomorphic imbedding of the graphs iff it is an isomorphic imbedding of the corresponding metric spaces. QED

Proof of the Corollary. In view of the reduction used in the proof of the main theorem, the clique problem is equivalent to the problem of finding an e-distinct set of n elements in $M(G)$. QED

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